Supplementary file

New seismic wave model for tight reservoirs: Incorporating non-Darcy flow and fractional viscoelasticity

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Appendix A: Plane-wave analysis of the ND-FV model

Applying Helmholtz's theorem, the seismic wave field can be decomposed into a longitudinal field (the P-wave, irrotational) and a transverse field (the S-wave, solenoidal) (Aki and Richards, 2002):

$$\mathbf{u} = \nabla \varphi_s + \nabla \times \mathbf{\Psi_s},\ \mathbf{U} = \nabla \varphi_f + \nabla \times \mathbf{\Psi_f} \tag{S1}$$

where φ_s, φ_f are scalar potential functions, and $\mathbf{\Psi_s}$, $\mathbf{\Psi_f}$ are vector potential functions. By substituting Eq. (S1) into Eq. (14), the wave field is separated into the P-wave field, which satisfies the following equation:

$$\begin{cases} \left(\psi_1 + \frac{4}{3}\psi_2\right) * \frac{\partial}{\partial t}\left(\Delta\varphi_s\right) + F\frac{(\alpha-\phi)^2}{\phi}\Delta\varphi_s + F(\alpha-\phi)\Delta\varphi_f = \frac{\partial^2}{\partial t^2}\left(\rho_{11}\varphi_s + \rho_{12}\varphi_f\right) \\ -\frac{\eta}{\kappa}\phi^{n+1}\left|\frac{\partial}{\partial t}\left(\nabla\varphi_f + \nabla\times\mathbf{\Psi_f} - \nabla\varphi_s - \nabla\times\mathbf{\Psi_s}\right)\right|^{n-1}\frac{\partial}{\partial t}\left(\varphi_f - \varphi_s\right), \\ F(\alpha-\phi)\Delta\varphi_s + F\phi\Delta\varphi_f = \frac{\partial^2}{\partial t^2}\left(\rho_{12}\varphi_s + \rho_{22}\varphi_f\right) \\ +\frac{\eta}{\kappa}\phi^{n+1}\left|\frac{\partial}{\partial t}\left(\nabla\varphi_f + \nabla\times\mathbf{\Psi_f} - \nabla\varphi_s - \nabla\times\mathbf{\Psi_s}\right)\right|^{n-1}\frac{\partial}{\partial t}\left(\varphi_f - \varphi_s\right) \end{cases} \tag{S2}$$

and the S-wave field that satisfies the following equation:

$$\begin{cases} \psi_2 * \frac{\partial}{\partial t}\left(\Delta\mathbf{\Psi_s}\right) = \frac{\partial^2}{\partial t^2}\left(\rho_{11}\mathbf{\Psi_s} + \rho_{12}\mathbf{\Psi_f}\right) - \frac{\eta}{\kappa}\phi^{n+1}\left|\frac{\partial}{\partial t}\left(\nabla\varphi_f + \nabla\times\mathbf{\Psi_f} - \nabla\varphi_s - \nabla\times\mathbf{\Psi_s}\right)\right|^{n-1}\frac{\partial}{\partial t}\left(\mathbf{\Psi_f} - \mathbf{\Psi_s}\right) \\ 0 = \frac{\partial^2}{\partial t^2}\left(\rho_{12}\mathbf{\Psi_s} + \rho_{22}\mathbf{\Psi_f}\right) + \frac{\eta}{\kappa}\phi^{n+1}\left|\frac{\partial}{\partial t}\left(\nabla\varphi_f + \nabla\times\mathbf{\Psi_f} - \nabla\varphi_s - \nabla\times\mathbf{\Psi_s}\right)\right|^{n-1}\frac{\partial}{\partial t}\left(\mathbf{\Psi_f} - \mathbf{\Psi_s}\right) \end{cases} \tag{S3}$$

For convenience in calculating the phase velocity and inverse quality factor, Eqs. (S2) and (S3) are transformed from the space-time domain to the wavenumber-frequency domain (Dvorkin and Nur, 1993; Aki and Richards, 2002; Carcione, 2022). Suppose that the wave propagates in an isotropic medium along x direction. Wave equation Eq. (14) has plane-wave solutions in the following form (Dvorkin and Nur, 1993; Zhang et al., 2022):

$$\mathbf{u} = \left[u, 0, 0\right]^T = \left[u_0 \exp^{i(kx-\omega t)}, 0, 0\right]^T, \mathbf{U} = \left[U, 0, 0\right]^T = \left[U_0 \exp^{i(kx-\omega t)}, 0, 0\right]^T \tag{S4}$$

where u_0 and U_0 are the initial amplitudes of the solid and fluid, respectively; k is the wavenumber, and ω is the angular frequency.

Based on the assumption that the porous medium is statistically isotropic and the initial amplitudes are constants, Eq. (S2) has plane-wave solutions, which can be written down as (Aki and Richards, 2002; Evans, 2010):

$$\varphi_s = \tilde{\varphi}_s \exp^{i(kx-\omega t)}, \varphi_f = \tilde{\varphi}_f \exp^{i(kx-\omega t)} \tag{S5}$$

where $\tilde{\varphi}_s$ and $\tilde{\varphi}_f$ are constants. Applying the divergence operation to Eq. (S1) and then performing the Fourier transform yields the following relationship between $\tilde{\varphi}_s, \tilde{\varphi}_f, u_0$, and U_0:

$$\frac{\tilde{\varphi}_f}{\tilde{\varphi}_s} = \frac{U_0}{u_0} = \frac{-M^*Y - F\dfrac{\alpha(\alpha-\phi)}{\phi}Y + \rho_1}{F\alpha Y - \rho_2} \tag{S6}$$

where $Y = (k/\omega)^2, \rho_1 = \rho_{11} + \rho_{12}, \rho_2 = \rho_{12} + \rho_{22}$, and $M*$ represents the viscoelastic matrix modulus (Evans, 2010), defined as:

$$\begin{aligned} M^* &= M_1^* + \frac{4}{3}M_2^* \\ M_1^* &= -i\omega\mathcal{F}[\psi_1] = K_m\left[\frac{1+(-i\omega)^{\xi}\tau_{\varepsilon_1}}{1+(-i\omega)^{\xi}\tau_{\sigma_1}}\right] \\ M_2^* &= -i\omega\mathcal{F}[\psi_2] = \mu\left[\frac{1+(-i\omega)^{\beta}\tau_{\varepsilon_2}}{1+(-i\omega)^{\beta}\tau_{\sigma_2}}\right] \end{aligned} \tag{S7}$$

where $\mathcal{F}[\cdot]$ denotes Fourier transform. Substituting the plane-wave solutions in Eq. (S5) into the Lamé function Eq. (S2) yields the following equation:

$$\mathbf{A\varphi = 0} \tag{S8}$$

where $\boldsymbol{\varphi} = [\widetilde{\varphi_s}, \widetilde{\varphi_f}]^T$ and

$$\mathbf{A} = \begin{bmatrix} -M^*Y - F\dfrac{\alpha(\alpha-\phi)}{\phi}Y + \rho_1 & -F\alpha Y + \rho_2 \\ -F(\alpha-\phi)Y + \rho_{12} - i\dfrac{\eta\phi^{n+1}|\omega|^{n-1}}{\omega\kappa}|U_0 - u_0|^{n-1} & -F\phi Y + \rho_{22} + i\dfrac{\eta\phi^{n+1}|\omega|^{n-1}}{\omega\kappa}|U_0 - u_0|^{n-1} \end{bmatrix} \tag{S9}$$

Assuming that Eq. (S8) has nontrivial solutions, the determinant of the coefficient matrix **A** should be equal to zero. Thus, the nonlinear equation for Y given in Eq. (17) is obtained. By solving it numerically, the root is obtained and denoted as Y_P. Substituting Y_P into Eq. (15) yields the phase velocity V_P and inverse quality factor Q_P^{-1} of the P-wave.

Similarly, Eq. (S3) has plane-wave solutions (Aki and Richards, 2002; Evans, 2010):

$$\boldsymbol{\Psi}_{\mathbf{s}} = \tilde{\boldsymbol{\Psi}}_{\mathbf{s}} \exp^{i(kx-\omega t)}, \boldsymbol{\Psi}_{\mathbf{f}} = \tilde{\boldsymbol{\Psi}}_{\mathbf{f}} \exp^{i(kx-\omega t)} \tag{S10}$$

where $\tilde{\boldsymbol{\Psi}}_{\mathbf{s}}$ and $\tilde{\boldsymbol{\Psi}}_{\mathbf{f}}$ are constant vectors. Substituting the plane-wave solutions in Eq. (S10) into the wave equation in Eq. (S3) yields:

$$\mathbf{B}\boldsymbol{\Psi} = \mathbf{0} \tag{S11}$$

where $\boldsymbol{\Psi} = \left[\tilde{\boldsymbol{\Psi}}_{\mathbf{s}}, \tilde{\boldsymbol{\Psi}}_{\mathbf{f}}\right]^T$ and

$$\mathbf{B} = \begin{bmatrix} -M_2^* Y + \rho_1 & \rho_2 \\ -\rho_{12} + i\dfrac{\eta\phi^{n+1}\left|\omega\right|^{n-1}}{\omega\kappa}|U_0 - u_0|^{n-1} & -\rho_{22} - i\dfrac{\eta\phi^{n+1}\left|\omega\right|^{n-1}}{\omega\kappa}|U_0 - u_0|^{n-1} \end{bmatrix} \tag{S12}$$

To ensure that Eq. (S11) has nontrivial solutions, the determinant of $\mathbf{B}$ should equal to zero. Therefore, Eq. (19) gives the nonlinear equation for Y, and its numerical solution yields Y_S. Substituting Y_S into Eq. (16) yields the phase velocity V_S and the inverse quality factor Q_S^{-1} of the S-wave.

Appendix B: Plane-wave analysis of the ND and FV models

Applying plane-wave analysis, following derivations similar to those presented for the ND-FV model in Appendix A, yields the expressions for the phase velocities and inverse quality factors of the P- and S-waves for the ND model (Eq. (5)) and for a friction-free FV model based on Eq. (9), in which the frictional dissipation term is omitted to emphasize the intrinsic viscoelastic effect.

For the P-wave, the dispersion equation calculated by the ND model (Eq. (5)) is:

$$a_5 Y^2 + a_4 Y + a_0 = \frac{ib_1}{\omega}|\,\omega\,|^{n-1}\left|\frac{c_1 - c_6 Y}{c_3 - c_4 Y}\right|^{n-1}(c_1 - c_6 Y) \tag{S13}$$

where

$$\begin{aligned} a_5 &= -\left(K_m + \frac{4}{3}\mu\right)F\phi,\ \ c_6 = K_m + \frac{4}{3}\mu + F\frac{\alpha^2}{\phi}, \\ a_4 &= F\phi\rho_{11} - 2F\left(\alpha - \phi\right)\rho_{12} + \left(K_m + \frac{4}{3}\mu + F\frac{\left(\alpha - \phi\right)^2}{\phi}\right)\rho_{22} \end{aligned} \tag{S14}$$

The numerical solution of Eq. (S1) gives $Y_{\text{ND-P}}$. The phase velocity $V_{\text{ND-P}}$ and the inverse quality factor $Q_{\text{ND-P}}^{-1}$ of the P-wave calculated by the ND model (Eq. (5)) are (Carcione, 2022; Zhang et al., 2022):

$$V_{\text{ND-P}} = \frac{1}{\operatorname{Re}\left(\sqrt{\mathrm{Y}_{\text{ND-P}}}\right)}, \quad Q_{\text{ND-P}}^{-1} = \left| \frac{2\operatorname{Im}\left(\sqrt{Y_{\text{ND-P}}}\right)}{\operatorname{Re}\left(\sqrt{Y_{\text{ND-P}}}\right)} \right| \tag{S15}$$

For the plane-wave analysis of the S-wave, the following nonlinear function of Y is obtained:

$$a_6 Y + a_0 = \frac{ib_1}{\omega} |\omega|^{n-1} \left| \frac{c_1 - c_7 Y}{c_3} \right|^{n-1} \left(c_1 - c_7 Y\right) \tag{S16}$$

where $a_6 = \mu\rho_{22}$, $c_7 = \mu$, and the detailed expressions of a_0, b_1, c_1, c_3 are found in Eq. (18). The numerical solution of Eq. (S16) gives $Y_{\text{ND-S}}$. The phase velocity $V_{\text{ND-S}}$ and the inverse quality factor $Q_{\text{ND-S}}^{-1}$ of the S-wave calculated by the ND model (Eq. (5)) are (Carcione, 2022):

$$V_{\text{ND-S}} = \frac{1}{\operatorname{Re}\left(\sqrt{\mathrm{Y}_{\text{ND-S}}}\right)}, \quad Q_{\text{ND-S}}^{-1} = \left| \frac{2\operatorname{Im}\left(\sqrt{Y_{\text{ND-S}}}\right)}{\operatorname{Re}\left(\sqrt{Y_{\text{ND-S}}}\right)} \right| \tag{S17}$$

For the P-wave, the dispersion equation calculated by the friction-free FV model is:

$$a_2 Y^2 + a_1 Y + a_0 = 0 \tag{S18}$$

The detailed expressions of a_0, a_1, a_2 are found in Eq. (18). For the friction-free FV model, the dispersion equation in Eq. (S18) is quadratic in Y, and its solutions can be written explicitly. The phase velocity $V_{\text{FV-P}}$ and the inverse quality factor $Q_{\text{FV-P}}^{-1}$ of the P-wave calculated by the friction-free FV model are (Carcione, 2022):

$$V_{\text{FV-P}} = \frac{1}{\operatorname{Re}\left(\sqrt{\mathrm{Y}_{\text{FV-P}}}\right)}, \quad Q_{\text{FV-P}}^{-1} = \left| \frac{2\operatorname{Im}\left(\sqrt{Y_{\text{FV-P}}}\right)}{\operatorname{Re}\left(\sqrt{Y_{\text{FV-P}}}\right)} \right| \tag{S19}$$

where

$$Y_{\text{FV-P}} = \frac{-a_1 - \sqrt{{a_1}^2 - 4a_2 a_0}}{2a_2} \tag{S20}$$

The phase velocity $V_{\text{FV-S}}$ and the inverse quality factor $Q_{\text{FV-S}}^{-1}$ of the S-wave calculated by the friction-free FV model are:

$$V_{\text{FV-S}} = \frac{1}{\operatorname{Re}\left(\sqrt{\dfrac{\rho_{11}\rho_{22} - {\rho_{12}}^2}{M_2^* \rho_{22}}}\right)}, \quad Q_{\text{FV-S}}^{-1} = \left| \frac{2\operatorname{Im}\left(\sqrt{\dfrac{\rho_{11}\rho_{22} - {\rho_{12}}^2}{M_2^* \rho_{22}}}\right)}{\operatorname{Re}\left(\sqrt{\dfrac{\rho_{11}\rho_{22} - {\rho_{12}}^2}{M_2^* \rho_{22}}}\right)} \right| \tag{S21}$$

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