

Original article

Design of Gaussian-modulated asymptotic solution networks for flow in shale gas reservoirs without using labeled data and grid

Xiang Li¹, Qian Wang¹, Daolun Li¹^{*}, Wenshu Zha¹, Xinan Zhang², Lei He¹

¹*School of Mathematics, Hefei University of Technology, Hefei 230601, P. R. China*

²*Graduate School of Arts and Sciences, Columbia University, New York, NY 10027, USA*

Keywords:

Shale gas reservoirs
nonlinear flow modeling
physics-driven neural networks
asymptotic solution networks
Gaussian modulation factor

Cited as:

Li, X., Wang, Q., Li, D., Zha, W., Zhang, X., He, L. Design of Gaussian-modulated asymptotic solution networks for flow in shale gas reservoirs without using labeled data and grid. *Advances in Geo-Energy Research*, 2026, 21(2): 139-153.
<https://doi.org/10.46690/ager.2026.08.05>

Abstract:

Shale gas flow equations are characterized by source-sink terms, adsorption terms and pressure-dependent fluid properties, resulting in strong nonlinearity. To the best of our knowledge, no existing deep learning study has solved shale gas flow equations without labeled data and grid. To address this gap, this study proposes an asymptotic solution network with a Gaussian modulation factor. The proposed network combines an asymptotic network, constructed from the asymptotic solution of a well in a homogeneous infinite-acting reservoir, with a correction network to correct the output of the asymptotic network. A Gaussian modulation factor is further introduced to enable the neural network to concentrate on regions with large pressure gradients and strong nonlinearity. Additionally, linear interpolation is employed to decrease the learning difficulty caused by the nonlinearity of the gas formation volume factor and viscosity. Numerical results demonstrate that the solutions of the proposed network closely match numerical simulations, demonstrating the effectiveness of the proposed model. Ablation studies reveal that the Gaussian modulation factor significantly accelerates convergence and improves the solution accuracy.

1. Introduction

Deep learning has achieved significant breakthroughs across numerous scientific and engineering fields, leveraging the exceptional capability of neural networks to mine complex data patterns and approximate functions (E et al., 2017; Asadollahpour et al., 2024; Liu et al., 2025; Gabry et al., 2026; Gómez-Camperos et al., 2026). This progress has injected fresh momentum into interdisciplinary research and engineering innovation. Extensive theoretical studies have demonstrated that deep neural networks exhibit highly efficient and accurate properties in approximating complex functions and nonlinear systems, thereby offering a novel paradigm for tackling problems that are challenging to solve by traditional numerical methods (Alpak et al., 2023; Xue et al., 2023; Kast

and Hesthaven, 2024; Ko and Park, 2025).

Solving partial differential equations (PDEs) lies at the core of scientific computing and has long relied on traditional numerical techniques such as finite difference and finite element methods (Adeleye and Akanji, 2022; Xu et al., 2023; Li et al., 2024). However, when confronted with high-dimensional problems, complex geometries, strong nonlinearities, or multi-scale couplings, these methods often require fine grids and meticulous discretization (Zheng and Li, 2022; Busto et al., 2023; Eskandari et al., 2025). This means prohibitive computational costs, slow convergence rates, and potential numerical instabilities. In recent years, physics-informed neural networks (PINNs) and their variants have emerged as a promising solution that incorporates physical laws as constraints

(Raissi et al., 2019). By embedding the governing equations and initial/boundary conditions into the loss function, they present a flexible and potentially powerful alternative for solving such complex PDEs (Jagtap et al., 2020; Meng et al., 2020; Niaki et al., 2021). PINNs can be broadly categorized into two paradigms: Discretization-based and continuous-form approaches.

For discretization-based approaches, the governing equations are discretized numerically and then solved by a neural network. These approaches fall into two main categories: Finite volume methods and finite element methods. The finite volume approach combines convolutional neural networks with discretization schemes and incorporates physics-informed constraints to enhance generalization, and has been extended to two-phase flow problems (Long et al., 2019; Zhang et al., 2023; Zhu et al., 2025). The finite element approach constructs the loss function based on finite element discretization to circumvent the difficulties of automatic differentiation in the presence of large gradients (Wang et al., 2025a). These methods ensure physical consistency via established discretization, yet their accuracy and efficiency depend on mesh quality. As they cannot leverage the grid-free advantage of neural networks, increasing attention has turned to methods that are free of grids and discretization.

For continuous-form approaches, the PDEs are solved without grids or numerical discretization. The loss is constructed from the governing equation residuals and evaluated via automatic differentiation of the neural network outputs, thereby reducing discretization errors and enhancing flexibility. Variational formulations of the equation residuals have been introduced to integrate initial conditions, boundary conditions and data mismatch into a unified optimization framework, leading to improved solution accuracy, particularly in data-scarce scenarios (Liu and Wu, 2023). To further enhance physical consistency, gradient-enhanced PINNs incorporate residual gradient information into the loss function, thereby imposing stronger constraints on the governing equations (Yu et al., 2022). More recently, these concepts have been extended to wave propagation problems by explicitly enforcing boundary conditions, resulting in improved physical fidelity and predictive accuracy (Xie et al., 2024). However, existing continuous-form neural network approaches and PINNs variants have been mainly validated on relatively simple PDEs, including linear or weakly nonlinear cases, while their application to seepage equations with strong nonlinearity, source/sink terms, and transient behavior remains challenging. Such methods often suffer from training instability, slow convergence, or even convergence failure when dealing with steep gradients or strong source/sink effects, limiting their ability to efficiently solve seepage equations.

Inspired by the PINNs concept and the analytical solution, Shen et al. (2023) first proposed asymptotic solution-based networks (ASNNets) to solve the single-phase oil flow equation, which incorporate prior analytical knowledge into the neural network architecture. Different from existing continuous-form approaches, ASNNets consist of an asymptotic network and a correction network. The asymptotic network is constructed from the analytical solution of a homogeneous infinite-acting

reservoir, providing a physics-informed approximation of pressure evolution and capturing the dominant flow behavior. The correction network further learns the deviations between the analytical assumptions and practical reservoir conditions, thereby improving the solution accuracy for complex reservoirs. This framework provides a new strategy for label-free seepage equation solving and enables continuous solution prediction without relying on grid-based discretization or labeled data. Building upon this, Wang et al. (2025b) introduced scaling before activation and gradient-enhanced constraints to extend the method to variable-rate conditions.

Despite the success of ASNNets in solving single-phase oil flow equations, their application to the more complex flow mechanisms of shale gas remains challenging. Shale gas reservoirs are typically characterized by low permeability, low porosity and adsorption effects (Cai et al., 2022; He et al., 2023; Wu et al., 2025; Zhang and Liu, 2025; Bouaffou et al., 2026; Sun et al., 2026). During production, gas undergoes adsorption and desorption within the pore space, and the amount of adsorbed gas changes with pressure, further increasing the nonlinearity of the equations. Meanwhile, the viscosity and formation volume factor of the gas change significantly with pressure, resulting in governing equations with stronger nonlinearities and multi-scale coupling characteristics. Furthermore, during tight reservoir development, dramatic pressure changes near the wellbore create a distinct pressure funnel with extremely large gradients (Lu et al., 2024). This makes it difficult for neural networks to converge stably, while it also reduces computational efficiency and solution accuracy.

To solve the single-phase shale gas flow equation with adsorption using a continuous-form approach, this paper proposes an ASNNets with Gaussian modulation (ASNNets-GM). The framework with dual-branch architecture is first designed based on the homogeneous infinite-acting reservoir line-source solution, consisting of an asymptotic network and a correction network. Next, to effectively capture the steep pressure gradients near the wellbore, a Gaussian modulation factor is developed to improve convergence stability and solving accuracy. Furthermore, to alleviate the strong nonlinearity induced by sharp pressure-dependent variations in gas viscosity and the formation volume factor, linear interpolation is introduced to approximate their pressure dependence, thereby simplifying the governing equations while maintaining physical consistency.

2. Pressure profile around a well and its Gaussian distribution

In order to investigate the pressure evolution during shale gas production and identify the challenges it poses for neural network-based solutions, the pressure distribution characteristics were analyzed. Fig. 1 illustrates the reservoir pressure distribution at a certain production stage, with the wellbore located at the coordinate origin. It can be observed that reservoir pressure changes differently at different locations: A sharp pressure change occurs near the wellbore, while pressure changes slowly when locations are far from it. This indicates a great pressure gradient and strong nonlinear characteristics

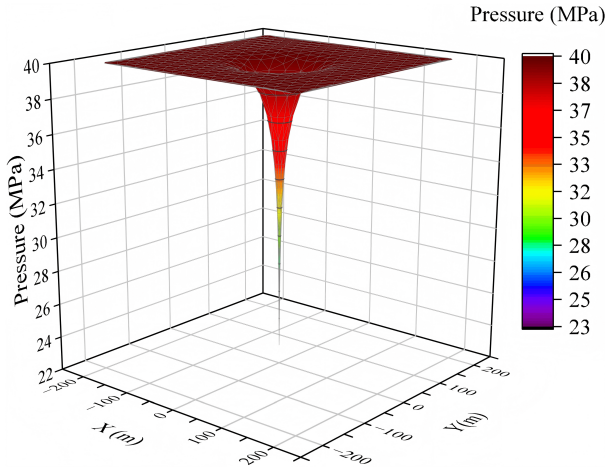


Fig. 1. Schematic of reservoir pressure distribution during production.

in the near-well region.

Such pressure distribution features pose significant challenges for neural networks in solving seepage equations. The high gradients and strong nonlinearity near the wellbore tend to induce gradient vanishing or oscillation, thereby degrading training stability and substantially slowing convergence. Meanwhile, conventional PINNs typically adopt a globally uniform residual weighting strategy during training, which makes it difficult to highlight critical information in the high-gradient near-wellbore region.

Therefore, during the solution process, more attention should be paid to the near-wellbore region by assigning higher weights that guide the neural network to preferentially learn and approximate the solution features in this region. Accordingly, a Gaussian function is introduced as a spatial weighting function, whose mathematical expression is given in (Lifshits, 1995):

$$G(x, y, |\tau, \sigma|) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left\{ -\frac{\left[\sqrt{(x-x_0)^2 + (y-y_0)^2} - \tau \right]^2}{2\sigma^2} \right\} \quad (1)$$

where x, y are the coordinates of any point in the plane; τ represents the mean; σ is a standard deviation parameter, which is optimized by deep learning; x_0, y_0 are the coordinates of the reference point.

The modulation characteristics under different parameter settings are illustrated in Fig. 2. The probability density reaches its maximum at the mean and decays monotonically with increasing distance. The Gaussian function inherently exhibits the property of central enhancement and far-field suppression, enabling smooth and continuous emphasis on local critical regions.

Based on this property, the local weighting mechanism of the Gaussian distribution is incorporated into the construction of the physics-informed neural network, yielding a weighting strategy consistent with the spatial characteristics of pressure distribution in shale gas reservoirs.

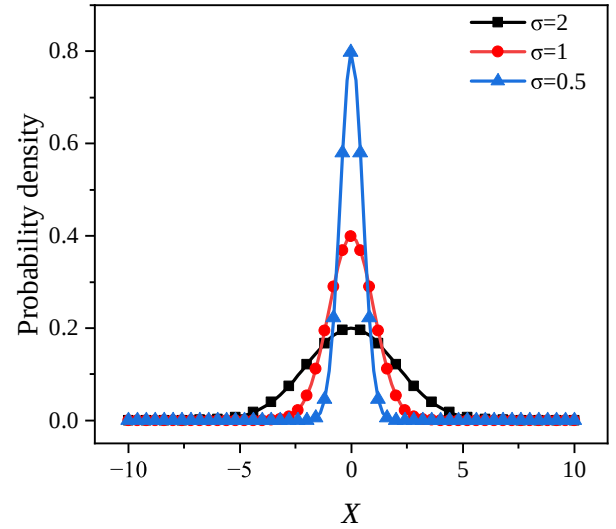


Fig. 2. Schematic diagram of Gaussian distribution with different σ values.

3. Single-phase gas seepage equation and asymptotic solution

The single-phase flow equation is derived by combining Darcy's law with the law of mass conservation. For the shale gas reservoirs with gas adsorption, the shale-gas flow model can be expressed as follows:

$$\nabla \cdot \left[\frac{k}{\mu B} (\nabla p) \right] = \frac{\partial}{\partial t} \left(\frac{\phi}{B} \right) + \frac{\partial}{\partial t} (\rho_s V_{ads}), \quad x, y \in \Omega_f, \quad t \in [0, T] \quad (2)$$

where k denotes permeability; μ denotes gas viscosity; B and ϕ denote the formation volume factor and porosity, respectively; ρ_s denotes the density of shale; Ω_f denotes the control domain.

For Eq. (2), the volume of absorbed gas is as follows:

$$V_{ads} = V_L \frac{p}{p + p_L} \quad (3)$$

where V_L represents the Langmuir adsorption constant, that is 0.002 m³/kg in this paper; p denotes reservoir pressure; p_L denotes Langmuir pressure, and is 7 MPa in this paper.

The initial and boundary conditions for the governing equation are:

$$\begin{aligned} p(x, y, t_0) &= p_0 \\ \frac{2\pi r_w h k}{\mu B} \frac{\partial p}{\partial r} &= Q, \quad r \in \partial\Omega_{IB} \\ \frac{\partial p}{\partial n} &= 0, \quad n \in \partial\Omega_{OB} \end{aligned} \quad (4)$$

where p_0 represents initial pressure; r denotes the distance from the wellbore to a given point in the formation; r_w denotes well radius; h denotes the effective thickness of the formation; Q denotes well production rate; Ω_{IB} and Ω_{OB} denote inner boundary and outer boundary, respectively.

Under constant flow conditions, based on the Ei function:

$$\text{Ei}(x) = \int_{-\infty}^x \frac{e^{-t}}{t} dt \quad (5)$$

The introduction of the Langmuir isotherm into the conti-

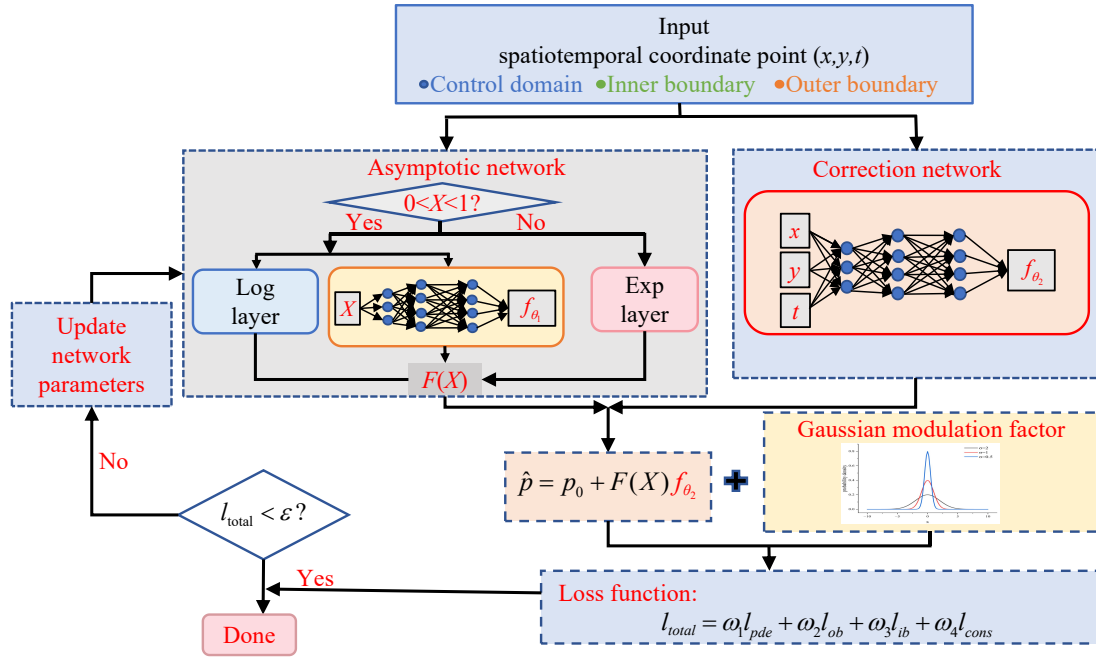


Fig. 3. Framework of ASNets-GM.

nuity equation for adsorbed gas makes the seepage equation strongly nonlinear. While a pseudo-pressure transformation allows for a linearized analytical solution, the pressure must be recovered by inverse interpolation from the pseudo-pressure. Given that the desorption effect is negligible at early times (as $t \rightarrow 0$), it can then be omitted. Therefore, the linear-source solution of the homogeneous infinite-acting reservoir can be expressed as:

$$p \approx p_0 - \frac{Q\mu}{4\pi kh} \text{Ei} \left[\frac{\mu \phi_{ref} (C_f + C_r)}{4k} \cdot \frac{(x - x_w)^2 + (y - y_w)^2}{t} \right] \quad (6)$$

where x_w and y_w denote the coordinates of the wellbore center; ϕ_{ref} denotes the reference porosity; C_r and C_f represent the fluid compressibility coefficient and rock compressibility coefficient, respectively.

The line-source solution describes the pattern of rapid early pressure decline near the wellbore and its continuous evolution over time. Thus, this analytical solution provides crucial physical insights and a theoretical foundation for constructing subsequent related solution methods.

4. Method

4.1 Network structure

To better represent the heterogeneity, the argument of the Ei function in Eq. (6) is written as:

$$X(x, y, t) = \alpha \frac{(x - x_w)^2 + (y - y_w)^2}{t} \quad (7)$$

where α represents a learnable parameter.

It is well known that the asymptotic solution of Ei(X) is expressed as:

$$\text{Ei}(X) = \begin{cases} \gamma - \ln|X|, & 0 < X < 1 \\ \frac{e^{-X}}{X}, & X \geq 1 \end{cases} \quad (8)$$

According to Eqs. (6) and (8), when $t \rightarrow 0$, the pressure p rapidly converges to the initial condition p_0 at a rate of e^{-X}/X ; when $t \rightarrow \infty$, the rate of change of pressure p is $-\gamma - \ln X$, where γ is Euler's constant.

Since the actual reservoir is not a homogeneous infinite-acting reservoir, a network is required to correct for deviations caused by this assumption. The overall architecture of the ASNets-GM framework consists of two core modules: The asymptotic network f_{θ_1} and the correction network f_{θ_2} , as shown in Fig. 3. The solution of the shale-gas flow equation is expressed as:

$$\hat{p} = p_0 + F(X)f_{\theta_2} \quad (9)$$

where $\hat{p}$ denotes the pressure solved by the neural network; $F(X)$ denotes the output of asymptotic network.

The design of f_{θ_1} is inspired by the asymptotic solution described in Eq. (8) and is constructed to help $F(X)$ to approximate the function of Ei(X). Therefore, $F(X)$ is expressed as follows:

$$F(X) = \begin{cases} 0, & X \leq 0 \\ \beta_1 f_{\theta_1}(\eta, t) - \beta_2 \ln|X|, & 0 < X < 1 \\ \beta_3 \frac{e^{-X}}{X}, & X \geq 1 \end{cases} \quad (10)$$

where β_1 , β_2 , β_3 represent learnable parameters; $f_{\theta_1}(\eta, t)$ is the output values of neural networks.

The correction network f_{θ_2} is used to correct the error caused by the homogeneous infinite-acting reservoir assumption, and it further refines the output of the asymptotic network to enhance solution accuracy. The final results are presented below:

$$\hat{p} = p_0 + F(X)f_{\theta_2}$$

$$= \begin{cases} p_0, & X \leq 0 \\ p_0 + f_{\theta_2} [\beta_1 f_{\theta_1}(\eta, t) - \beta_2 \ln |X|], & 0 < X < 1 \\ p_0 + f_{\theta_2} \beta_3 \frac{e^{-X}}{X}, & X \geq 1 \end{cases} \quad (11)$$

Eq. (11) shows that when $t \rightarrow 0$, $X \rightarrow \infty$ is known from Eq. (6), and thus $e^{-X}/X \rightarrow 0$. Finally, $\hat{p}$ is equal to p_0 . This means that the initial conditions are hard-coded into the neural network.

4.2 Gaussian modulation factor

To enhance the ability of the model to capture local gradient variations, a Gaussian modulation factor is incorporated into this framework, resulting in the proposed ASNets-GM, which is formulated as follows:

$$\hat{p} = p_0 + G(x, y, \tau, t)F(X)f_{\theta_2}$$

$$= \begin{cases} p_0, & X \leq 0 \\ p_0 + G(x, y, \tau, t)f_{\theta_2} [\beta_1 f_{\theta_1}(\eta, t) - \beta_2 \ln(|X|)], & 0 < X < 1 \\ p_0 + G(x, y, \tau, t)f_{\theta_2} \beta_3 \frac{e^{-X}}{X}, & X \geq 1 \end{cases} \quad (12)$$

The Gaussian modulation factor enables the neural network to more easily learn locally sharp pressure variations, thereby improving both the solution speed and accuracy.

4.3 Linear interpolation for pressure-dependent gas properties

During shale gas reservoir development, continuous pressure depletion leads to obvious variations in the gas pressure-volume-temperature properties, particularly the formation volume factor B and gas viscosity μ , which exhibit strong nonlinear dependence on pressure. Under low-permeability and adsorption-dominated flow conditions, this nonlinearity is further amplified, posing significant challenges to the training of PINNs:

$$pv = nRT$$

$$B = \frac{v_{res}}{v_{sc}} \quad (13)$$

$$\mu = f(\rho_g)$$

where v denotes the volume of gas; n represents the amount of substance; R denote ideal gas constant; T denotes thermodynamic temperature; v_{res} denotes the volume under reservoir conditions; v_{sc} is the volume under standard conditions; ρ_g denotes the density of shale gas.

In order to effectively reduce problem complexity while preserving physical consistency, this study employs a linear interpolation approach to approximate the pressure-dependent gas formation volume factor and viscosity. This treatment smooths the relationship between pressure and fluid properties, significantly alleviating the strong nonlinearity in the governing flow equations and thereby improving training stability. In reservoir numerical simulation, the pressure-volume-temperature properties are provided in tabular form, and the viscosity and formation volume factor corresponding to

a given pressure are obtained by interpolation (Ertekin et al., 2001). In the analytical approach, however, the relevant pressure-volume-temperature properties are expressed in terms of explicit formulas.

4.4 Loss function

The incorporation of the mass conservation equation as a physical constraint significantly enhances the numerical stability during network training. This constraint not only suppresses oscillations and accelerates iterative convergence but also explicitly embeds physical laws into the optimization process, guiding the solution toward globally optimal states consistent with realistic flow behavior.

Although the deep learning-based solution of the seepage equation does not require any grid information, a control volume must be constructed around each sampled point to enforce mass conservation. Since the perpendicular bisection (PEBI) gridding method is available, a PEBI grid is adopted to partition the sampled points prior to mass conservation calculations.

For a sampled point $(x^i, y^i, \hat{p}^i)$, it has a control volume V . The time domain is discretized into a sequence of time $\{t_s\}_{s=1}^{N_t-1}$. Within each adjacent time interval $[t_s, t_{s+1}]$, the pressure variation in every control volume must satisfy the mass conservation condition, following the specific form of the mass balance equation below:

$$\sum_{i=1}^N V^i \left[\frac{\phi(p_{t_{s+1}}^i)}{B(p_{t_{s+1}}^i)} - \frac{\phi(p_{t_s}^i)}{B(p_{t_s}^i)} \right] + V_{ads}\rho_s = Q(t_{s+1} - t_s) \quad (14)$$

where N denotes the number of grid blocks.

Therefore, the loss function obtained based on the mass conservation equation is as follows:

$$l_{cons} = \left\| Q(t_{s+1} - t_s) - V_{ads}\rho_s - \sum_{i=1}^N V^i \left[\frac{\phi(\hat{p}_{t_{s+1}}^i)}{B(\hat{p}_{t_{s+1}}^i)} - \frac{\phi(\hat{p}_{t_s}^i)}{B(\hat{p}_{t_s}^i)} \right] \right\|_2 \quad (15)$$

where the l_{cons} is the loss terms of the mass conservation.

Combined with the control equation, and the inner and outer boundary conditions, the following loss function is constructed:

$$l_{pde} = \left\| \nabla \cdot \left[\frac{k}{\mu B} (\nabla \hat{p}) \right] - \frac{\partial}{\partial t} \left(\frac{\phi}{B} \right) - \frac{\partial}{\partial t} (\rho_s V_{ads}) \right\|_2 \quad (16)$$

$$l_{ib} = \left\| \frac{2\pi r_w h k}{\mu B} \frac{\partial \hat{p}}{\partial r} - Q \right\|_2 \quad (17)$$

$$l_{ob} = \left\| \frac{\partial \hat{p}}{\partial n} \right\|_2 \quad (18)$$

where the l_{pde} , l_{ib} and l_{ob} are the loss terms of the control equation, the inner boundary condition and outer boundary condition, respectively.

In conclusion, the loss function of the shale-gas seepage equation is constructed as follows:

$$l_{total} = \omega_1 l_{pde} + \omega_2 l_{ob} + \omega_3 l_{ib} + \omega_4 l_{cons} \quad (19)$$

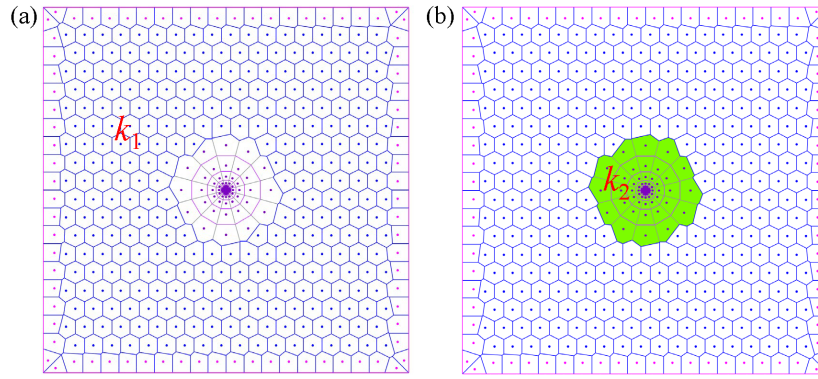


Fig. 4. Schematic diagrams of (a) homogeneous and (b) heterogeneous reservoir.

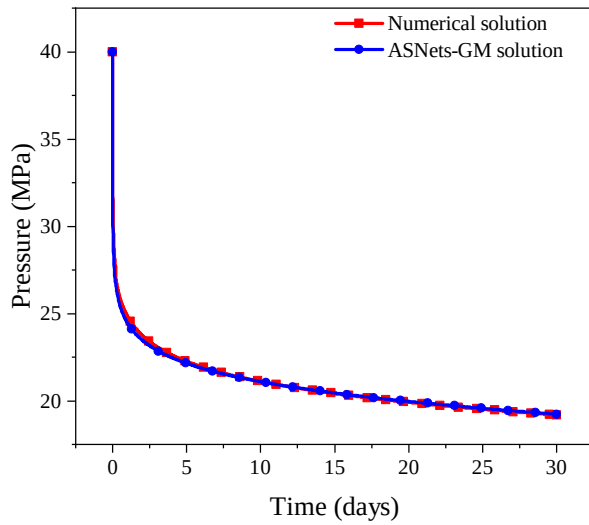


Fig. 5. BHP curves of homogeneous reservoir from ASNets-GM and numerical solutions.

where the l_{total} denotes the total loss; ω_1 , ω_2 , ω_3 and ω_4 are adjustable weighting parameters.

The boundary conditions are imposed as soft constraints. Their weighting coefficients are treated as hyperparameters and manually tuned based on the loss evolution, convergence behavior, computational accuracy, and efficiency. How to automatically adjust the parameters in the loss function remains one of the future research directions.

5. Simulation results

This study conducts numerical analyses on rectangular homogeneous and heterogeneous shale gas reservoirs. The reservoir area is $400 \times 400 \text{ m}^2$, with a production well located at the center as shown in Fig. 4. Fig. 4(a) shows the homogeneous shale gas reservoir and Fig. 4(b) shows the heterogeneous shale gas reservoir. The well is produced at a constant rate of $Q = 4,800 \text{ m}^3/\text{d}$ for 30 days. For the homogeneous case, the reservoir parameters are listed in Table 1. The heterogeneous case is constructed by setting the permeability to $k_2 = 0.014 \text{ mD}$ within a 50 m radius around the wellbore, while keeping all other parameters identical to those of the homogeneous case.

Table 1. Relevant parameters of the homogeneous reservoir.

Wellbore radius (m)	Reservoir thickness (m)	Porosity (-)	Permeability (mD)	Rock compressibility (Pa^{-1})
0.1	10	0.05	0.009	1.5×10^{-9}

In order to obtain high-accuracy reference solutions, a reservoir numerical simulator based on PEBI grids is employed. The simulator performs spatial discretization using the control volume method and solves the governing equations with a fully implicit time integration scheme. This numerical tool has been continuously improved by our group since 2009, validated against several commercial reservoir simulators, and successfully applied in multiple oilfields (Li et al., 2025). Given a computational domain of 30 days in time and a control domain and the outer boundaries in space, the combined number of temporal and spatial sampling points is 60,000. Considering the importance of pressure gradients in the vicinity of the wellbore, the inner boundary region has 8,000 high-density sampling points to enhance the ability of the model to capture variations in bottom-hole pressure (BHP).

In this study, the network hyperparameters of ASNets-GM, including the number of hidden layers and neurons, are determined based on empirical design and preliminary experiments to achieve a balance between convergence behavior, training stability, as well as computational accuracy and efficiency. During the model training stage, all cases adopt the Adam optimizer with a unified learning rate of 10^{-3} to balance computational efficiency and training stability. The asymptotic network adopts a 10-layer residual architecture with a tanh activation function to progressively approximate the reservoir dynamics, while the correction network consists of a 3-layer residual structure with a Leaky ReLU activation function to finely adjust the output of the asymptotic network.

5.1 Homogeneous reservoir case

The ASNets-GM model converges within only 58 training iterations, and all network parameters are updated using the loss function defined in Eq. (19). The calculated BHP over a 30-day production period is obtained by ASNets-GM in Fig. 5. The red solid line denotes the numerical solution, while the

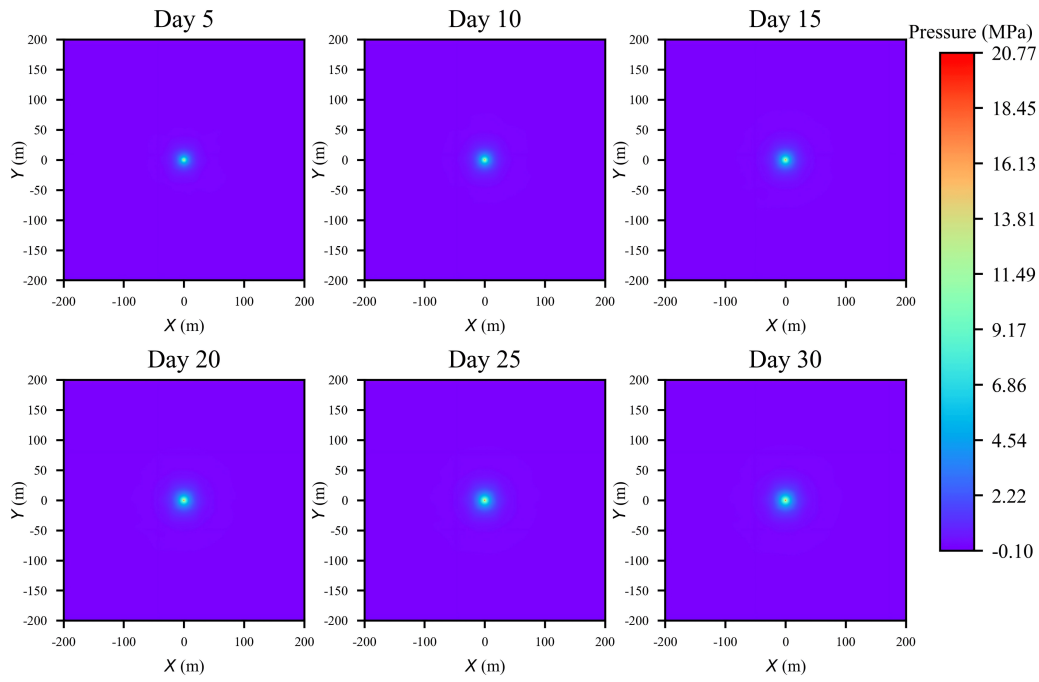


Fig. 6. Numerical solutions of homogeneous reservoir pressure drawdown distribution at 5-30 days.

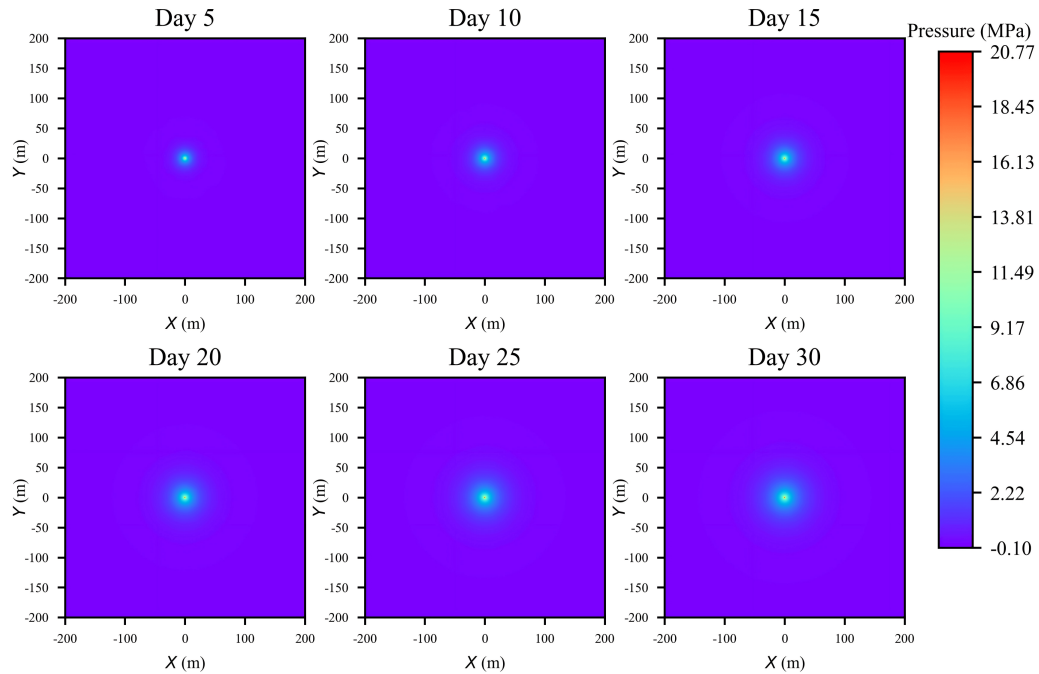


Fig. 7. Homogeneous reservoir pressure drawdown distributions at 5-30 days obtained by ASNets-GM.

blue dashed line represents the neural network solution, and the two are in close agreement. This shows that ASNets-GM can solve the shale-gas flow equation with high accuracy.

In order to evaluate the capability of ASNets-GM in characterizing the evolution of reservoir pressure, the pressure drawdown distributions at 5, 10, 15, 20, 25, and 30 days of production were obtained using both numerical simulations and ASNets-GM. The corresponding results are shown in Figs. 6 and 7, respectively. With the gas continuously flowing toward the wellbore during production, the pressure drawdown

increases in the near-well region. Meanwhile, the pressure disturbance gradually propagates outward into the surrounding reservoir, resulting in steep pressure gradients near the wellbore. The solutions of ASNets-GM are consistent with the numerical results and successfully reproduce the evolution of the pressure field during production, demonstrating its capability to capture the pressure propagation characteristics.

In order to quantitatively assess the solution accuracy, the relative L_2 error distributions between ASNets-GM solutions and numerical solutions were further evaluated in Fig. 8. The

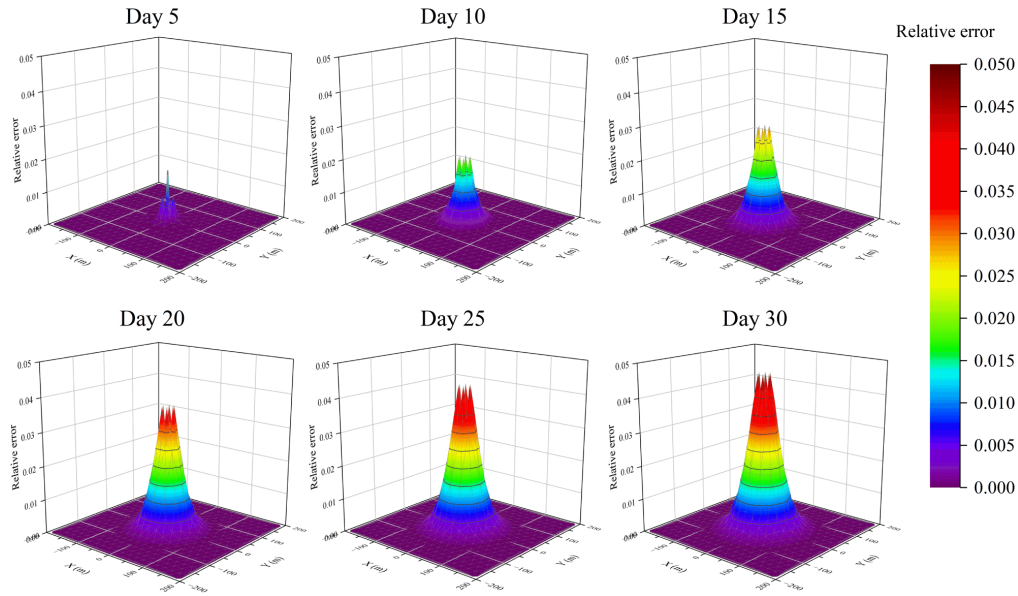


Fig. 8. Relative L_2 error distributions between numerical and ASNets-GM solutions at 5-30 days in a homogeneous reservoir.

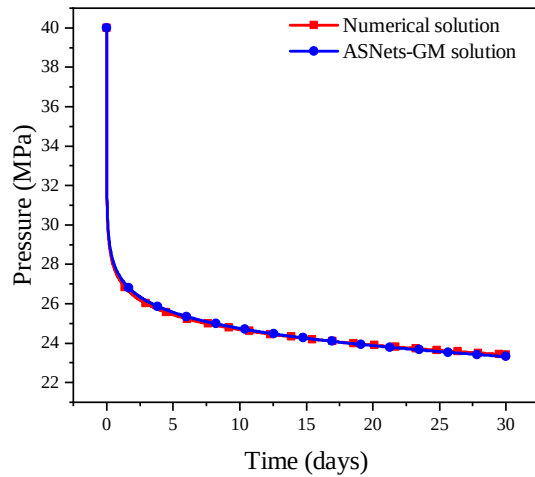


Fig. 9. BHP curves of heterogeneous reservoir from ASNets-GM and numerical solutions.

maximum relative error remains below 0.04 over the entire production period. The error is mainly concentrated in regions with large pressure gradients near the wellbore, where the strong nonlinear coupling of adsorption effects and pressure-dependent gas properties introduces solution difficulty. Benefiting from the Gaussian modulation factor, the network assigns greater attention to regions with rapid pressure variations, enabling more effective learning of the near-wellbore flow characteristics. Despite the strong nonlinearity and rapid pressure variations in this region, ASNets-GM maintains high accuracy, demonstrating that the calculated results are in agreement with the numerical solutions, particularly in resolving the pressure gradients around the wellbore. These results demonstrate that the proposed model effectively captures both the dynamic evolution and fine-scale features of the reservoir pressure field. It also significantly alleviates the strong nonlinearity of adsorption-driven flow equations and enables efficient, stable, and physically meaningful solutions.

5.2 Heterogeneous reservoir case

In order to verify the applicability of the proposed model under more complex conditions, a heterogeneous reservoir case was further conducted. The ASNets-GM network achieved stable convergence after 65 training iterations, and BHP over a 30-day production period under heterogeneous reservoir conditions solved by the ASNets-GM model. The red solid line in Fig. 9 represents the numerical solution, while the blue dashed line denotes the model solution. The results demonstrate good agreement in both global trends and local details, indicating that ASNets-GM can accurately capture the pressure decline behavior even under heterogeneous reservoir conditions.

In order to further evaluate the capability of ASNets-GM in characterizing reservoir pressure evolution under heterogeneous permeability conditions, the pressure drawdown distributions at 5, 10, 15, 20, 25, and 30 days of production were obtained using both numerical simulations and ASNets-GM, as shown in Figs. 10 and 11, respectively. The overall pressure evolution trend under heterogeneous permeability conditions remains similar to that observed in homogeneous reservoirs. With production, the pressure drawdown gradually increases with decreasing distance from the wellbore, and the affected area expands. The solutions of ASNets-GM show high agreement with the numerical results and successfully reproduce the pressure evolution in heterogeneous reservoirs.

The relative L_2 error distributions between ASNets-GM solutions and numerical solutions were further analyzed to quantify the model accuracy under heterogeneous permeability conditions. The maximum relative error remains below 0.06 throughout the entire production period as shown in Fig. 12. The errors are mainly concentrated in the regions near the wellbore and transition zones with permeability variations. The former is larger and primarily attributed to the rapid pressure gradient changes and strong nonlinear effects induced

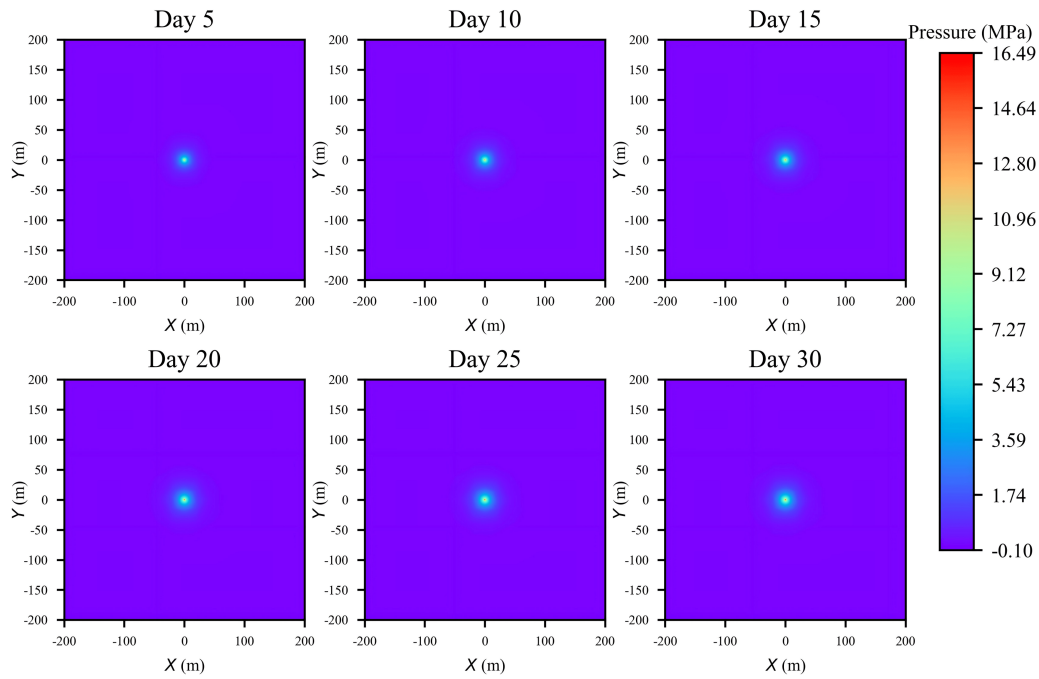


Fig. 10. Numerical solutions of heterogeneous reservoir pressure drawdown distribution at 5-30 days.

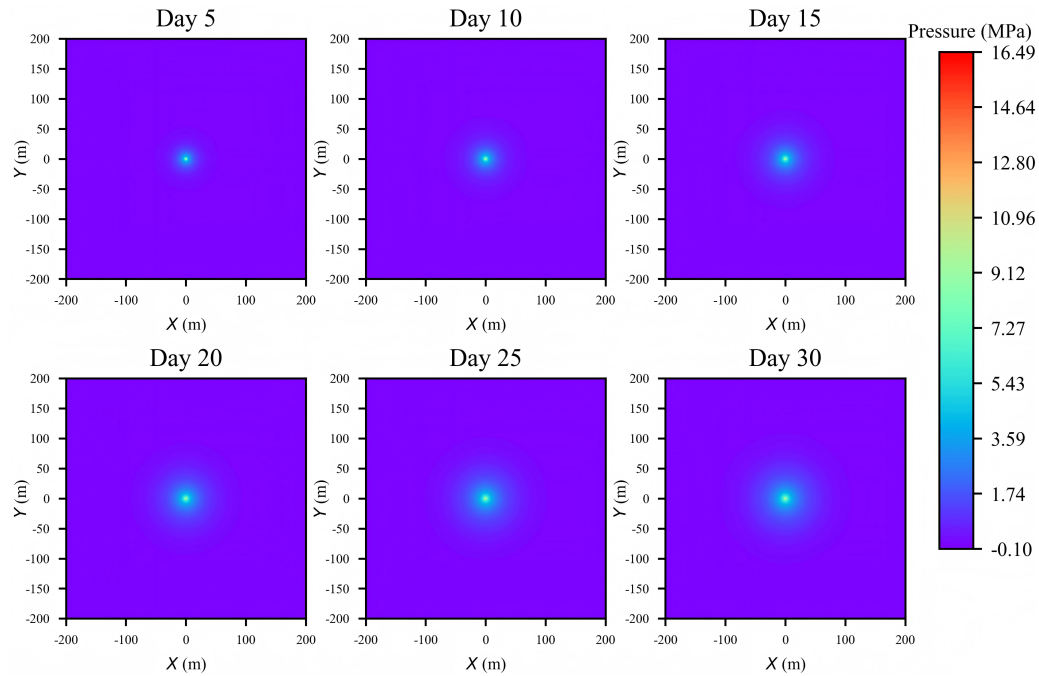


Fig. 11. Heterogeneous reservoir pressure drawdown distributions at 5-30 days obtained by ASNets-GM.

by adsorption gas, while the latter is small and originates from the complexity of local pressure gradients caused by spatial permeability variations. Benefiting from the Gaussian modulation factor, ASNets-GM enhances the learning of flow characteristics in the near-wellbore region. Although the solution errors remain in regions with permeability changes, the model achieves good agreement with the numerical solutions, demonstrating its effectiveness in solving heterogeneous problems.

5.3 Comparative analysis and ablation studies

Experimental analysis was conducted based on pressure prediction results under heterogeneous reservoir conditions, including both ablation studies and comparative experiments. These experiments were designed to verify the superiority of the proposed model.

In the ablation studies, all training configurations, including the optimizer and learning parameters, were kept identical, while specific components of the model, namely the asymp-

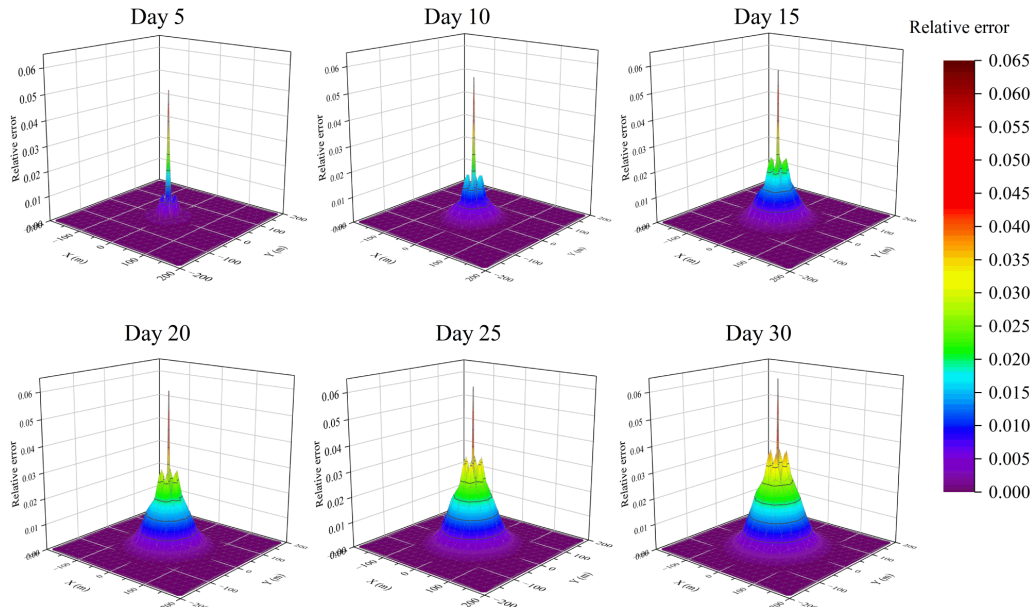


Fig. 12. Relative L_2 error distributions between numerical and ASNets-GM solutions at 5–30 days in a heterogeneous reservoir.

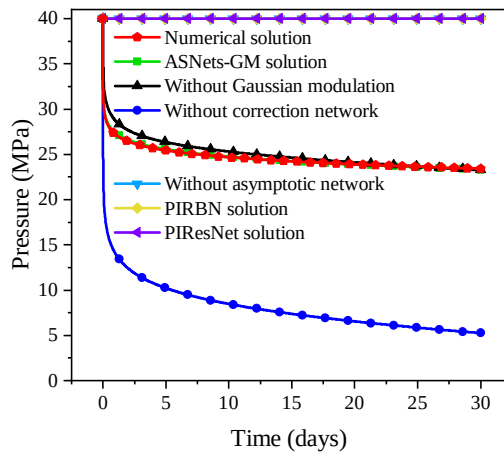


Fig. 13. BHP comparison of different models in a heterogeneous reservoir.

otic network, the correction network, and the Gaussian modulation factor, were removed individually. In particular, removing both the Gaussian modulation factor and the linear interpolation for gas parameters from ASNets-GM recovers the standard ASNets (Shen et al., 2023). By systematically analyzing the changes in model performance under different structural reductions, the contribution of each key module to the overall capability of solving the flow problem was quantitatively assessed.

For the comparative study, the proposed ASNets-GM model was evaluated against two representative physics-informed approaches, physics-informed radial basis network (PIRBN) (Bai et al., 2023) and PINNs combined with residual networks (PIResNet). All models were trained under the same computational settings, optimization strategies, and convergence criteria to ensure the fairness and comparability of the results. A total of seven ablation and comparative experiments were conducted (Fig. 13). By comparing the bottom-

hole pressures obtained from each case, the accuracy of the computational results under different conditions was validated.

The results show that the solutions of PIRBN, PIResNet and the model without asymptotic network are nearly horizontal pressure profiles and coincide with one another, as shown by the purple line in Fig. 13. It is obvious to that none of them is correct, which means that PIRBN and PIResNet cannot solve the shale-gas flow equation. Shale-gas flow equation has source/sink term and gas adsorption, which is a great challenge for deep learning to solve without label data. Special efforts are needed to tackle this problem, which is the aim of this paper.

As to the failure of the model without the asymptotic network, the reason is also understandable. The ASNets-GM network is constructed based on the asymptotic solution of the E_i function, which includes the solution as time tends to zero. This is equivalent to embedding the physical mechanism into the network. Without the asymptotic network - i.e., with no physical mechanism embedded - the neural network suffers from poor solution accuracy.

When the Gaussian modulation factor is removed, the model is able to capture the overall trend of BHP evolution, as shown by the black line in Fig. 13. However, noticeable deviations appear during the early-time period. This observation indicates that the Gaussian modulation factor significantly improves the ability of the model to resolve sharp pressure variations near the wellbore, particularly in regions characterized by steep pressure gradients.

When the correction network is removed, the model obtains incorrect BHP, as shown by the blue line in Fig. 13. The correction network is used to correct the error introduced by the homogeneous infinite-acting reservoir assumption. Without it, the analytical solution derived under the assumption is directly applied to the rectangular reservoir, resulting in significant errors.

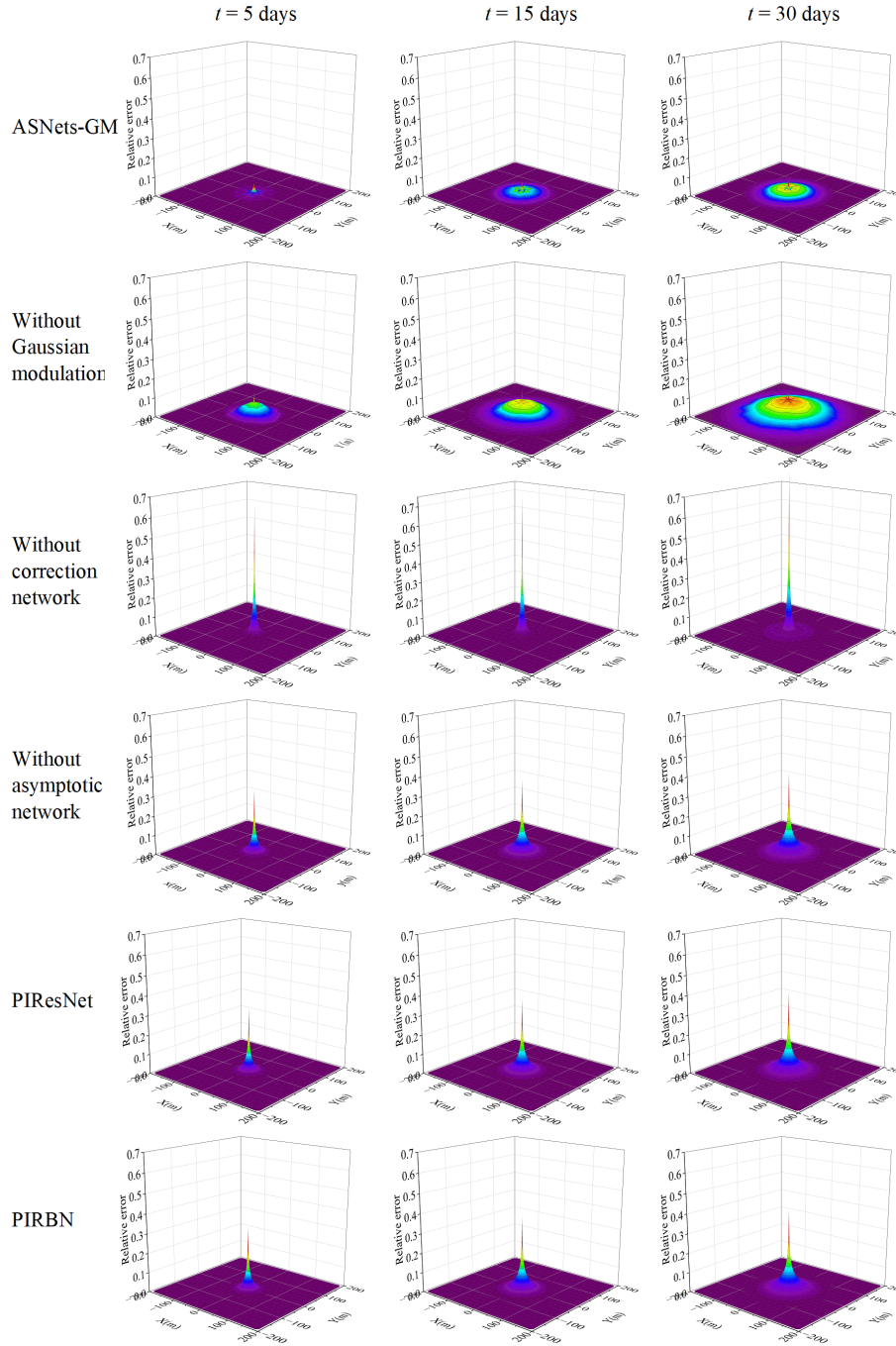


Fig. 14. Relative L_2 error distributions of pressure in a heterogeneous reservoir.

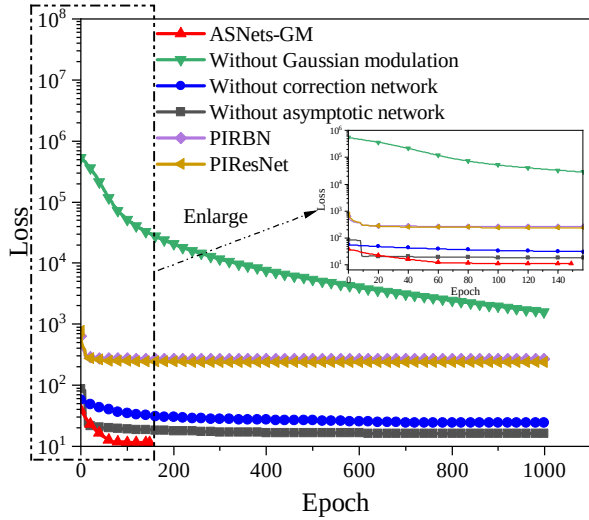
In order to provide a more intuitive comparison of the overall solution performance of each model, the L_2 relative error distributions of the proposed model, three ablation variants, PIRBN, and PIResNet at 5, 15, and 30 days were compared (Fig. 14). In terms of the maximum L_2 relative error, ASNets-GM achieved the best performance, with the error consistently remaining below 0.1, followed closely by ASNets-GM without the Gaussian modulation factor. In contrast, all other models exhibited maximum errors exceeding 0.2, with the largest exceeding 0.7.

In order to further quantitatively evaluate the solution

accuracy of different models, the average L_2 relative errors of the pressure distributions were compared within 50 m of the wellbore after 5, 15 and 30 days of production in Table 2. ASNets-GM achieved an average relative error on the order of 10^{-2} , whereas PIRBN and PIResNet exhibited errors on the order of 10^{-1} , indicating bad solution accuracy. Ablation studies revealed that removing the Gaussian modulation factor increases the relative L_2 error by approximately 2 to 3 times. Removing either the correction network or the asymptotic network further increases the error to about 5 times that of ASNets-GM. These results indicate that the combination of

Table 2. Average relative L_2 errors of pressure distribution within 50 m of the wellbore in a heterogeneous reservoir.

Time (days)	ASNets-GM	Without Gaussian modulation	Without correction network	Without asymptotic network	PIRBN	PIResNet
5	0.0198	0.0580	0.1400	0.1050	0.1050	0.1050
15	0.0256	0.0694	0.1700	0.1380	0.1380	0.1380
30	0.0317	0.0784	0.1950	0.1630	0.1630	0.1630

**Fig. 15.** Training iterations to convergence under heterogeneous conditions.

the asymptotic network, correction network and Gaussian modulation factor is key to achieving high solution accuracy. Notably, the relative L_2 errors of the model without the asymptotic network, PIRBN and PIResNet are identical, because all three lack prior physical constraints and thus converge to the initial pressure field, failing to capture the pressure drawdown induced by production. These differences arise from the same mechanisms discussed in the analysis of Fig. 13 and are therefore not repeated here.

In order to compare the training convergence of different models, all models except ASNets-GM were trained for 1,000 epochs. The convergence criterion was defined as a change in the total loss of less than 0.002. The loss evolution with training epochs were compared to show their different convergence speed for each model under heterogeneous reservoir conditions in Fig. 15. All models converge rapidly and plateau quickly. ASNets GM shows the lowest initial loss, as is evident from the enlarged view. The epoch counts for ASNets GM, the no Gaussian modulation variant, the no asymptotic network variant, the no correction network variant, PIRBN, and PIResNet are 65, 783, 56, 137, 19, and 121, respectively. Notably, although the latter four models require only moderate training epochs, their final solutions are erroneous, indicating that early convergence does not guarantee correctness.

When Gaussian modulation factor is neglected, the solution is rather good, as shown in Fig. 13. With nearly equivalent solution accuracy, ASNets-GM requires only $1/12.5 \approx 65/783$ training epochs to reach convergence than the model without

the Gaussian modulation factor. Therefore, the Gaussian modulation factor can improve the speed of convergence more than 10 times, demonstrating its good performance.

In terms of computational efficiency, the training times required for the six models under heterogeneous reservoir conditions over 1,000 epochs are presented in Table 3. ASNets-GM requires only 136.2 s. The case without the Gaussian modulation factor requires 1,784 s, approximately 16 times that of ASNets-GM. The cases without the correction network or the asymptotic network need less time compared with ASNets-GM. However, the solution is not correct, which can be seen from Fig. 13. PIRBN and PIResNet not only require approximately 26 and 4 times the training time of ASNets-GM but also the solution is incorrect. Overall, ASNets-GM achieves significantly higher accuracy with only a modest increase in computational cost, attributable to the Gaussian modulation factor, which accelerates convergence, and the asymptotic network, which improves solution accuracy.

6. Sensitivity tests

In order to further evaluate the applicability of ASNets-GM under different conditions, sensitivity analyses were conducted for permeability, production rate, and sampling density. In each analysis, the other parameters were fixed while only the target parameter was varied, whose influence on the model performance was evaluated. The permeability values were set to 0.002, 0.010 and 0.050 mD, while the production rates were selected as 2,800, 4,800 and 5,800 m^3/d .

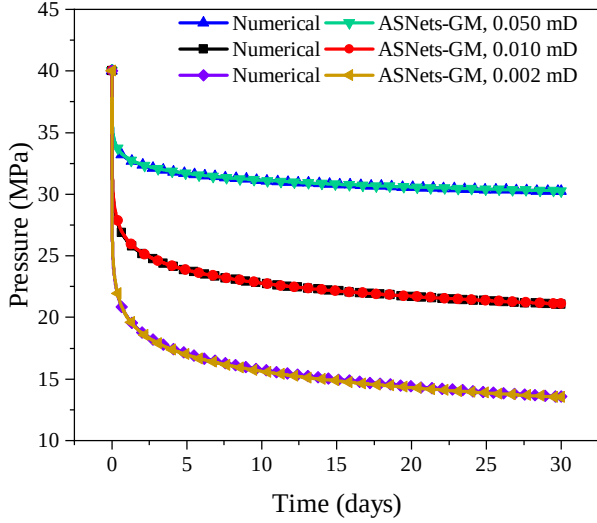
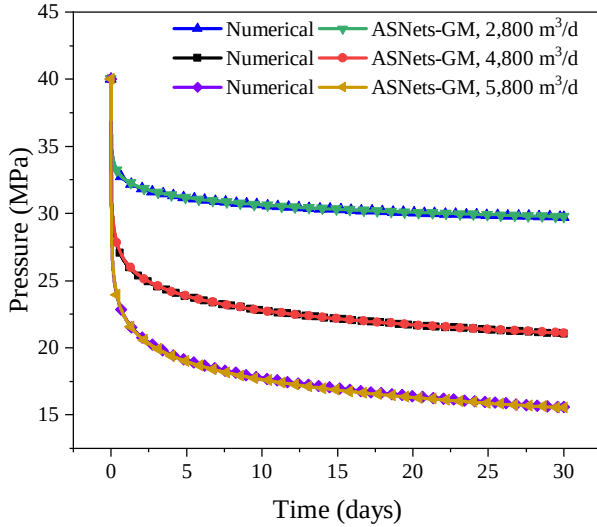
For the sampling density analysis, sampling was performed within the spatiotemporal computational domain corresponding to a 30-day production period. The baseline sampling strategy consisted of approximately 60,000 spatiotemporal sampling points. To enhance the capability of the network of learning flow characteristics in the near-wellbore region with high pressure gradients, 8,000 distributed sampling points were assigned around the wellbore. This sampling strategy was defined as the baseline sampling density δ . Furthermore, two additional sampling densities, 0.5δ and 2δ , were considered to investigate the effect of sampling point density on the solution performance of ASNets-GM.

6.1 Permeability

Permeability is a key reservoir parameter affecting fluid transport capacity in low-permeability shale reservoirs, whose variation directly affects pressure propagation and production behavior. Therefore, a permeability sensitivity analysis was conducted under a fixed production rate of $Q = 4,800 \text{ m}^3/\text{d}$

Table 3. Comparison of training times.

Model	ASNNets-GM	Without Gaussian modulation	Without correction network	Without asymptotic network	PIRBN	PIResNet
Training time (s)	136.2	2,194	123.4	116.2	3,627	563

**Fig. 16.** BHP comparison between numerical and ASNNets-GM solutions.**Fig. 17.** BHP comparison between numerical and ASNNets-GM solutions.

and with the sampling density fixed, denoted as δ , with permeability values of $k = 0.002$, 0.010 , and 0.050 mD, to evaluate the applicability of ASNNets-GM under different permeability conditions. It could be observed that the BHP decreases as permeability decreases (Fig. 16). Under a constant-rate production condition, reduced permeability increased flow resistance, requiring a larger pressure drawdown to maintain the same production rate, resulting in a lower stabilized BHP during the later production stage. For all permeability

cases, the ASNNets-GM solutions showed good agreement with the numerical solutions, demonstrating that the model can accurately capture the influence of permeability variations on reservoir pressure responses and remains robust for solving seepage problems under different permeability conditions.

6.2 Production rate

The production rate is another critical operational parameter that directly controls pressure depletion and reservoir pressure response during shale gas production. Therefore, a flow-rate sensitivity analysis was conducted while keeping the permeability $k = 0.010$ mD and sampling density δ fixed, with production rates of $Q = 2,800$, $4,800$, and $5,800$ m³/d, to evaluate the robustness of ASNNets-GM under different production rates. The BHP was different over the 30-day production period for different flow-rate cases, as shown in Fig. 17. The numerical solutions (red solid lines) and ASNNets-GM solutions (blue dashed lines) showed good agreement. It could be observed that the BHP decreases with increasing production rate. Under constant permeability conditions, a higher production rate requires a larger pressure drawdown to provide sufficient pressure gradient for fluid transport, resulting in a lower BHP during production. For all flow-rate cases, ASNNets-GM accurately captures the pressure response induced by production-rate variations, demonstrating its robustness in solving seepage problems under different production conditions.

6.3 Sampling density

The sampling density is an important factor affecting the computational efficiency and solution accuracy of neural network-based methods for solving PDEs. Therefore, a sampling density sensitivity analysis was conducted over a 30-day production period while keeping the permeability $k = 0.010$ mD and production rate $Q = 4,800$ m³/d unchanged. The baseline sampling density was denoted as δ , containing approximately 60,000 spatiotemporal sampling points, including 8,000 densely distributed sampling points in the high-gradient region near the wellbore. Two additional sampling densities, 0.5δ and 2δ , were further considered to investigate the influence of sampling point variations on the solution performance of ASNNets-GM. Table 4 summarizes the training time and average relative L_2 error of BHP under different sampling densities. The results indicate that increasing the sampling density increases the computational cost while providing more spatiotemporal information to enhance the capability of the network to characterize flow features, thereby slightly reducing the average relative L_2 error. However, once the sampling density is sufficient to describe the dominant flow characteristics,

Table 4. Training time and average relative L_2 error of ASNets-GM for different sampling densities.

Sampling density	Training time (s)	The average relative L_2 errors
0.5 δ	79.73	0.0235
1 δ	82.41	0.0232
2 δ	83.08	0.0231

further increasing the number of sampling points only provides limited improvement in prediction accuracy.

7. Conclusions

In this work, the ASNets-GM has been developed for the label-free solution of nonlinear single-phase shale gas flow equations with adsorption effects. The proposed framework adopts a dual-branch architecture inspired by the homogeneous infinite-acting line-source solution, in which an asymptotic network provides a solution for the homogeneous infinite-acting reservoir case, while a correction network accounts for deviations arising from complex conditions such as reservoir heterogeneity and reservoir boundaries. This architecture enables a stable and accurate representation of the pressure field without labeled data.

To address the great nonlinearity and steep pressure gradients near the wellbore, the Gaussian modulation factor has been introduced to adaptively weight the residual constraints. This strategy significantly improves convergence behavior and solution accuracy in regions with sharp gradients, where conventional physics-informed models often fail.

Overall, the proposed ASNets-GM demonstrates enhanced robustness and efficiency in solving shale gas flow equations. Nevertheless, the current framework still relies on manually tuned hyperparameters and is limited to single-phase flow. Therefore, future research will focus on adaptive hyperparameter optimization and the extension of the proposed framework to multiphase flow simulations.

Acknowledgements

This work was supported by the National Natural Science Foundation of China (Nos. 12372244 and 12172115).

Conflicts of interest

The authors declare no competing interest.

Open Access This article is distributed under the terms and conditions of the Creative Commons Attribution (CC BY-NC-ND) license, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

References

Adeleye, J. O., Akanji, L. T. A quantitative analysis of flow properties and heterogeneity in shale rocks using computed tomography imaging and finite-element based simulation. *Journal of Natural Gas Science and Engineering*, 2022, 106: 104742.

Alpak, F. O., Vamaraju, J., Jennings, J. W., et al. Augmenting

deep residual surrogates with fourier neural operators for rapid two-phase flow and transport simulations. *SPE Journal*, 2023, 28(6): 2982-3003.

Asadollahpour, S., Jiang, Z., Lewis, H., et al. Deep learning for pore-scale two-phase flow: Modelling drainage in realistic porous media. *Petroleum Exploration and Development*, 2024, 51(5): 1301-1315.

Bai, J., Liu, G., Gupta, A., et al. Physics-informed radial basis network (PIRBN): A local approximating neural network for solving nonlinear partial differential equations. *Computer Methods in Applied Mechanics and Engineering*, 2023, 415: 116290.

Busto, S., Dumbser, M., Río-Martín, L. An Arbitrary-Lagrangian-Eulerian hybrid finite volume/finite element method on moving unstructured meshes for the Navier-Stokes equations. *Applied Mathematics and Computation*, 2023, 437: 127539.

Bouaffou, J.-M. K., Jiang, S., Konan, R. Y. H. Fine characterization of pore structure to study the migration and retention behavior of fracturing fluids in normal pressure shale gas reservoirs: A case study of Longmaxi formation in the Pengshui area. *SPE Journal*, 2026, 31(1): 375-398.

Cai, J., Tian, Z., Wood, D. A. Chapter One - Pore-scale characterization and fractal analysis for gas migration mechanisms in shale gas reservoirs, in *Sustainable Geoscience for Natural Gas Subsurface Systems*, edited by David A. Wood and Jianchao Cai, Gulf Professional Publishing, Cambridge, MA, pp. 1-27, 2022.

Ertekin, T., Abou-Kassem, J. H., King, G. R. *Basic Applied Reservoir Simulation*. Richardson, TX, USA, Society of Petroleum Engineers, 2001.

Eskandari, Z., Avazzadeh, Z., Khoshsiar Ghaziani, R., et al. Dynamics and bifurcations of a discrete-time Lotka-Volterra model using nonstandard finite difference discretization method. *Mathematical Methods in the Applied Sciences*, 2025, 48(7): 7197-7212.

E, W., Han, J., Jentzen, A. Deep learning-based numerical methods for high-dimensional parabolic partial differential equations and backward stochastic differential equations. *Communications in Mathematics and Statistics*, 2017, 5: 349-380.

Gabry, M., Ramadan, A., Soliman, M. Calibration of stimulated reservoir volume estimation using continuous wavelet transform and advanced deep learning with rate transient analysis: A case study from the marcellus shale. *SPE Journal*, 2026, 31(8): 5507-5527.

Gómez-Camperos, J. A., Ruiz-Díaz, C. M., Hernández-Cely, M. M., et al. Classification of dense-gas/liquid flow patterns in horizontal pipe through pressure gradient time series encoded into images and deep learning. *SPE Journal*, 2026, 31(4): 2212-2227.

He, Y., He, Z., Tang, Y., et al. Interwell fracturing interference evaluation in shale gas reservoirs. *Geoenergy Science and Engineering*. 2023, 231: 212337

Jagtap, A. D., Kharazmi, E., Karniadakis, G. E. Conservative physics-informed neural networks on discrete domains for conservation laws: Applications to forward and inverse problems. *Computer Methods in Applied Mechan-*

- ics and Engineering, 2020, 365: 113028.
- Kast, M., Hesthaven, J. S. Positional embeddings for solving PDEs with evolutionary deep neural networks. *Journal of Computational Physics*, 2024, 508: 112986.
- Ko, S., Park, S. VS-PINN: A fast and efficient training of physics-informed neural networks using variable-scaling methods for solving PDEs with stiff behavior. *Journal of Computational Physics*, 2025, 529: 113860.
- Li, D., Wang, Q., Zha, W., et al. Inversion of Multiple Reservoir Parameters Based on Deep Neural Networks Guided by Lagrange Multipliers. *SPE Journal*, 2025, 30(5): 2632-2652.
- Li, X., Huang, X., Yu, S., et al. Numerical study on the transient pressure response of the vug in carbonate reservoirs. *Petroleum Science and Technology*, 2024, 42(22): 3264-3283.
- Lifshits, M. A. *Gaussian Random Functions*. Dordrecht, Netherlands, Springer, 1995.
- Liu, C., Wu, H. A. A variational formulation of physics-informed neural network for the applications of homogeneous and heterogeneous material properties identification. *International Journal of Applied Mechanics*, 2023, 15(8): 2350065.
- Liu, Z., Gudala, M., Yan, B. Efficient optimization of coupled geothermal reservoir modeling and power plant off-design based on deep learning. *Advances in Geo-Energy Research*, 2025, 18(1): 84-98.
- Long, Z., Lu, Y., Dong, B. PDE-Net 2.0: Learning PDEs from data with a numeric-symbolic hybrid deep network. *Journal of Computational Physics*, 2019, 399: 108925.
- Lu, Z., Li, X., Che, M. Shale oil shut-in and flowback mechanism and optimization strategy. *Journal of Energy Resources Technology*, 2024, 146(10): 102901.
- Meng, X., Li, Z., Zhang, D., et al. PPINN: Parareal physics-informed neural network for time-dependent PDEs. *Computer Methods in Applied Mechanics and Engineering*, 2020, 370: 113250.
- Niaki, S. A., Haghighat, E., Campbell, T., et al. Physics-informed neural network for modelling the thermochemical curing process of composite-tool systems during manufacture. *Computer Methods in Applied Mechanics and Engineering*, 2021, 384: 113959.
- Raissi, M., Perdikaris, P., Karniadakis, G. E. Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations. *Journal of Computational Physics*, 2019, 378: 686-707.
- Shen, L., Li, L., Zha, W., et al. Physical Asymptotic-Solution nets: Physics-driven neural networks solve seepage equations as traditional numerical solution behaves. *Physics of Fluids*, 2023, 35(2): 023603.
- Sun, H., Zhang, A., Fang, J., et al. Seepage field construction and technical directions for efficient development of shale gas reservoirs: A case study of Fuling shale gas field, Sichuan Basin, China. *Petroleum Exploration and Development*, 2026, 53(3): 787-803.
- Wang, C., Guo, H., Yan, X., et al. Improved physics-informed neural networks for the reinterpreted discrete fracture model. *Journal of Computational Physics*, 2025a, 520: 113491.
- Wang, C., Zhao, Y., Wu, R., et al. Shale reservoir storage of hydrogen: Adsorption and diffusion on shale. *Fuel*, 2024, 357: 129919.
- Wang, Q., Li, D., Zha, W., et al. Scaled asymptotic solution nets for unlabeled seepage equation solutions with variable well flow. *Physics of Fluids*, 2025b, 37(1): 013607.
- Xie, G., Fu, B., Li, H., et al. A gradient-enhanced physics-informed neural networks method for the wave equation. *Engineering Analysis with Boundary Elements*, 2024, 166: 105802.
- Wu, M., Chang, X., Guo, Y., et al. Advances, challenges, and opportunities for hydraulic fracturing of deep shale gas reservoirs. *Advances in Geo-Energy Research*, 2025, 15(1): 1-4.
- Xu, R., Guo, T., Xue, X., et al. Numerical simulation of fracturing and imbibition in shale oil horizontal wells. *Petroleum Science*, 2023, 20(5): 2981-3001.
- Xue, L., Li, D., Dou, H. Artificial intelligence methods for oil and gas reservoir development: Current progresses and perspectives. *Advances in Geo-Energy Research*, 2023, 10(1): 65-70.
- Yu, J., Lu, L., Meng, X., et al. Gradient-enhanced physics-informed neural networks for forward and inverse PDE problems. *Computer Methods in Applied Mechanics and Engineering*, 2022, 393: 114823.
- Zhang, K., Liu, S. Compressibility of unconventional gas shale formations: Implications for hydrogen geo-storage. *Gas Science and Engineering*, 2025, 142: 205715.
- Zhang, Z., Yan, X., Liu, P., et al. A physics-informed convolutional neural network for the simulation and prediction of two-phase Darcy flows in heterogeneous porous media. *Journal of Computational Physics*, 2023, 477: 111919.
- Zheng, Z., Li, X. Theoretical analysis of the generalized finite difference method. *Computers & Mathematics with Applications*, 2022, 120: 1-14.
- Zhu, H., Chen, Z., Gao, X., et al. A Physics-informed adaptive-wavelet neural network (PIAWNN) for reservoir simulation with two-phase flow. *Advances in Water Resources*, 2025, 207: 105174.